

Variational existence theory for hydroelastic solitary waves

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Abstract

This paper presents an existence theory for solitary waves at the interface between a thin ice sheet (modelled using the Cosserat theory of hyperelastic shells) and an ideal fluid (of finite depth and in irrotational motion) for sufficiently large values of a dimensionless parameter γ . We establish the existence of a minimiser of the wave energy \mathcal{E} subject to the constraint $\mathcal{I} = 2\mu$, where \mathcal{I} is the horizontal impulse and $0 < \mu \ll 1$, and show that the solitary waves detected by our variational method converge (after an appropriate rescaling) to solutions of the nonlinear Schrödinger equation with cubic focussing nonlinearity as $\mu \downarrow 0$.

Résumé

Une théorie variationnelle d'existence d'ondes solitaires hydroélastiques. Cette note présente une théorie d'existence d'ondes solitaires à l'interface entre une couche de glace mince (modélisée par la théorie des coques hyperélastiques de Cosserat) et un fluide parfait (de profondeur finie et irrotationnel), pour des valeurs suffisamment grandes d'un paramètre sans dimension γ . Nous montrons l'existence d'un minimiseur de l'énergie \mathcal{E} de l'onde sous la contrainte $\mathcal{I} = 2\mu$, où \mathcal{I} représente l'impulse horizontale et $0 < \mu \ll 1$. Nous démontrons que les ondes solitaires trouvées par notre méthode variationnelle convergent (après un changement d'échelle approprié) vers des solutions de l'équation de Schrödinger cubique focalisante, lorsque $\mu \downarrow 0$.

1. Introduction

1.1. The hydrodynamic problem

In this article we consider the two-dimensional irrotational flow of a perfect fluid beneath a thin ice sheet modelled using the Cosserat theory of hyperelastic shells (Plotnikov and Toland [8]). The fluid is bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = h + \eta(x, t)\}$; there is no cavitation between this surface and the ice sheet. The mathematical problem is to find an Eulerian velocity potential ϕ which satisfies the equations

$$\phi_{xx} + \phi_{yy} = 0, \quad 0 < y < 1 + \eta, \quad (1)$$

$$\phi_y = 0, \quad y = 0, \quad (2)$$

$$\phi_y = \eta_t + \phi_x \eta_x, \quad y = 1 + \eta, \quad (3)$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + \eta + \gamma H(\eta) = 0, \quad y = 1 + \eta \quad (4)$$

with

$$H(\eta) = \frac{1}{(1 + \eta_x^2)^{1/2}} \left[\frac{1}{(1 + \eta_x^2)^{1/2}} \left(\frac{\eta_{xx}}{(1 + \eta_x^2)^{5/2}} \right) \right]_x + \frac{1}{2} \left(\frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \right)^2$$

(see Guyenne and Parau [5]). Here we have introduced dimensionless variables, choosing h as length scale and $(h/g)^{1/2}$ as time scale; the parameter γ is defined by the formula $\gamma = \mathcal{D}/(\rho g h^4)$, where \mathcal{D} , ρ and

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g are respectively the coefficient of flexural rigidity for the ice sheet, the density of the fluid and the acceleration due to gravity. *Solitary hydroelastic waves* are non-trivial solutions of these equations of the form $\eta(x, t) = \eta(x + \nu t)$, $\phi(x, y, t) = \phi(x + \nu t, y)$ with $\eta(x + \nu t) \rightarrow 0$ as $x + \nu t \rightarrow \pm\infty$.

Equations (1)–(4) admit the conserved quantities

$$\mathcal{E}(\eta, \Phi) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\Phi G(\eta) \Phi + \eta^2 + \gamma \frac{\eta_{xx}^2}{(1 + \eta_x^2)^{5/2}} \right) dx, \quad \mathcal{I}(\eta, \Phi) = \int_{-\infty}^{\infty} \eta_x \Phi dx$$

(‘energy’ and ‘impulse’) associated with translation invariance in t and x ; the *Dirichlet-Neumann operator* $G(\eta)$ is defined by $G(\eta) = (1 + \eta_x^2)^{1/2} \phi_n|_{y=1+\eta}$, in which ϕ is the harmonic function in $0 < y < 1 + \eta$ with $\phi_y|_{y=0} = 0$ and $\phi|_{y=1+\eta} = \Phi$. A hydroelastic solitary wave corresponds to a critical point of the energy under the constraint of fixed impulse (the potential ϕ is recovered from Φ by solving the above boundary-value problem) and therefore a critical point of the functional $\mathcal{E} - \nu \mathcal{I}$, where the Lagrange multiplier ν gives the wave speed. Proposition 1.1 confirms in particular that $\mathcal{E}, \mathcal{I} \in C^\infty(U \times H_\star^{1/2}(\mathbb{R}))$, where $U = B_M(0)$ is a neighbourhood of the origin in $H^2(\mathbb{R})$ and $H_\star^{1/2}(\mathbb{R}), H_\star^{-1/2}(\mathbb{R})$ are the completions of $\mathcal{S}(\mathbb{R})$ with respect to the norms $\|\eta\|_{\star, 1/2} := (\int_{-\infty}^{\infty} (1 + k^2)^{-1/2} k^2 |\hat{\eta}|^2 dk)^{1/2}$ and $\|\eta\|_{\star, -1/2}^2 := (\int_{-\infty}^{\infty} (1 + k^2)^{1/2} k^{-2} |\hat{\eta}|^2 dk)^{1/2}$.

Proposition 1.1 *The mapping $W^{1,\infty}(\mathbb{R}) \rightarrow \text{GL}(H_\star^{1/2}(\mathbb{R}), H_\star^{-1/2}(\mathbb{R}))$ given by $\eta \mapsto (\Phi \mapsto G(\eta)\Phi)$ is analytic at the origin.*

Restricting to small-amplitude waves, we seek minimisers of \mathcal{E} subject to the constraint $\mathcal{I} = 2\mu$, where μ is a small positive number, and establish the following theorem.

Theorem 1.1 *The following statements hold for each sufficiently large value of γ .*

- (i) *The set D_μ of minimisers of \mathcal{E} over $S_\mu = \{(\eta, \Phi) \in U \times H_\star^{1/2}(\mathbb{R}) : \mathcal{I}(\eta, \Phi) = 2\mu\}$ is non-empty and lies in $H^4(\mathbb{R}) \times H_\star^{1/2}(\mathbb{R})$. Furthermore, the estimate $\|\eta\|_2 \lesssim \mu^{1/2}$ holds uniformly over D_μ .*
- (ii) *Suppose that $\{(\eta_n, \Phi_n)\}$ is a minimising sequence for \mathcal{E} . There exists a sequence $\{x_n\} \subseteq \mathbb{R}$ with the property that a subsequence of $\{(\eta_n(x_n + \cdot), \Phi_n(x_n + \cdot))\}$ converges in $H^2(\mathbb{R}) \times H_\star^{1/2}(\mathbb{R})$ to a function in D_μ .*

Remark 1 (‘conditional energetic stability of the set of minimisers’) *Suppose that $(\eta, \Phi) : [0, T] \rightarrow U \times H_\star^{1/2}(\mathbb{R})$ is a solution to (1)–(4) in the sense that $\mathcal{E}(\eta(t), \Phi(t)) = \mathcal{E}(\eta(0), \Phi(0))$, $\mathcal{I}(\eta(t), \Phi(t)) = \mathcal{I}(\eta(0), \Phi(0))$ for all $t \in [0, T]$ (see Ambrose and Siegel [1] for a discussion of the initial-value problem). It follows from Theorem 1.1 that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\text{dist}((\eta(0), \Phi(0)), D_\mu) < \delta$ implies $\text{dist}((\eta(t), \Phi(t)), D_\mu) < \varepsilon$ for $t \in [0, T]$.*

1.2. Heuristics

The existence of small-amplitude solitary waves is predicted by studying the dispersion relation for the linearised version of (1)–(4). Linear waves of the form $\eta(x, t) = \cos k(x + \nu t)$ exist whenever $\nu = \nu(k)$, where $\nu(k)^2 = (1 + \gamma k^4)/f(k)$, $f(k) := |k| \coth |k|$. The function $k \mapsto \nu(k)$, $k \geq 0$ has a unique global minimum $\nu_0 = \nu_0(k_0)$ with $k_0 > 0$ (see Figure 1(a)). Note also that $g(k) := 1 + \gamma k^4 - \nu_0^2 f(k) \geq 0$ with equality precisely when $k = \pm k_0$, and solving the equation $g'(k_0) = 0$ yields the relationship $\gamma = \gamma_0(k_0)$, where $\gamma_0(k_0) = f'(k_0)(k_0^3(4f(k_0) - f'(k_0)))^{-1}$, so that γ_0 is a strictly monotone decreasing function of k_0 with $\lim_{k_0 \rightarrow 0} \gamma_0(k_0) = \infty$ and $\lim_{k_0 \rightarrow \infty} \gamma_0(k_0) = 0$.

Bifurcations of nonlinear solitary waves are expected whenever the linear group and phase speeds are equal, so that $\nu'(k) = 0$ (see Dias and Kharif [3, §3]). We therefore expect the existence of small-amplitude solitary waves with speed near ν_0 ; the waves bifurcate from a linear periodic wave train with frequency $k_0 \nu_0$ (see Figure 1(b)). The appropriate model equation for this type of solution is the cubic nonlinear Schrödinger equation

$$2iA_T - \frac{1}{4}g''(k_0)A_{XX} + \frac{3}{2}\left(\frac{1}{2}A_3 + A_4\right)|A|^2A = 0, \quad (5)$$

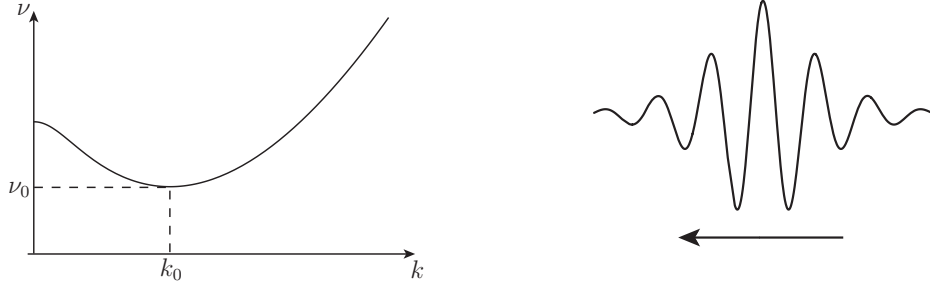


Figure 1. (a) Dispersion relation for linear hydroelastic waves. (b) Small-amplitude envelope solitary waves with speed $\nu = \nu_0 + 2(\nu_0 f(k_0))^{-1} \mu^2 \nu_{\text{NLS}}$ (where $\nu_{\text{NLS}} < 0$) predicted by nonlinear Schrödinger theory.

in which

$$\eta(x, t) = \frac{1}{2} \mu (A(X, T) e^{ik_0(x + \nu_0 t)} + \text{c.c.}) + O(\mu^2), \quad X = \mu(x + \nu_0 t), \quad T = 2k_0(\nu_0 f(k_0))^{-1} \mu^2 t$$

and the abbreviation ‘c.c.’ denotes the complex conjugate of the preceding quantity; the values of the constants A_3 and A_4 are $A_3 = -\frac{1}{3}g(2k_0)^{-1}(A_3^1)^2 - \frac{2}{3}g(0)^{-1}(A_3^2)^2$ and $A_4 = A_4^1 - \nu_0^2 A_4^3$, where

$$A_3^1 = \nu_0^2 f(2k_0) f(k_0) + \frac{1}{2} \nu_0^2 f(k_0)^2 - \frac{3}{2} k_0^2 \nu_0^2, \quad A_3^2 = \nu_0^2 f(k_0) + \frac{1}{2} \nu_0^2 f(k_0)^2 - \frac{1}{2} \nu_0^2 k_0^2,$$

$$A_4^1 = -\frac{5}{12} \gamma k_0^6, \quad A_4^2 = \frac{1}{6} f(k_0)^2 (f(2k_0) + 2) - \frac{1}{2} k_0^2 f(k_0).$$

Equation (5) was derived in the present context by Milewski and Wang [7, §2], who also noted that the coefficient $\frac{1}{2} A_3 + A_4$ is negative for sufficiently small values of k_0 , or equivalently for sufficiently large values of γ (corresponding to sufficiently shallow water in physical variables). At this level of approximation, a solution to equation (5) of the form $A(X, T) = e^{i\nu_{\text{NLS}} T} \zeta(X)$ with $\zeta(X) \rightarrow 0$ as $X \rightarrow \pm\infty$, so that ζ is a homoclinic solution of the ordinary differential equation

$$-\frac{1}{4} g''(k_0) \zeta_{xx} - 2\nu_{\text{NLS}} \zeta + \frac{3}{2} \left(\frac{1}{2} A_3 + A_4 \right) |\zeta|^2 \zeta = 0 \quad (6)$$

with $\nu_{\text{NLS}} = -\frac{9}{8} \alpha_{\text{NLS}}^2 g''(k_0)^{-1} \left(\frac{1}{2} A_3 + A_4 \right)^2$ and $\alpha_{\text{NLS}} = \frac{1}{2} \left(\frac{1}{4} \nu_0 f(k_0) + \frac{1}{8} \omega \right)^{-1}$, corresponds to a solitary wave with speed $\nu = \nu_0 + 2(\nu_0 f(k_0))^{-1} \mu^2 \nu_{\text{NLS}}$.

Proposition 1.2 Suppose that $\frac{1}{2} A_3 + A_4 < 0$. The set of complex-valued homoclinic solutions to the ordinary differential equation (6) is $D_{\text{NLS}} = \{e^{i\omega} \zeta_{\text{NLS}}(\cdot + y) : \omega \in [0, 2\pi), y \in \mathbb{R}\}$, where

$$\zeta_{\text{NLS}}(x) = \alpha_{\text{NLS}} \left(-3g''(k_0)^{-1} \left(\frac{1}{2} A_3 + A_4 \right) \right)^{\frac{1}{2}} \text{sech} \left(-3\alpha_{\text{NLS}} g''(k_0)^{-1} \left(\frac{1}{2} A_3 + A_4 \right) x \right).$$

Our second theorem confirms the heuristic argument given above.

Theorem 1.2 For each sufficiently small value of k_0 the set D_μ of minimisers of \mathcal{E} over S_μ satisfies

$$\sup_{(\eta, \Phi) \in D_\mu} \inf_{\omega \in [0, 2\pi], x \in \mathbb{R}} \|\zeta_\eta - e^{i\omega} \zeta_{\text{NLS}}(\cdot + x)\|_1 \rightarrow 0$$

as $\mu \downarrow 0$, where we write $\eta_1^+(x) = \frac{1}{2} \mu \zeta_\eta(\mu x) e^{ik_0 x}$ and $\eta_1^+ = \mathcal{F}^{-1}[\chi_{[k_0 - \delta_0, k_0 + \delta_0]} \hat{\eta}]$ with $\delta_0 \in (0, \frac{1}{3} k_0)$. Furthermore, the speed ν_μ of the corresponding solitary wave satisfies $\nu_\mu = \nu_0 + 2(\nu_0 f(k_0))^{-1} \nu_{\text{NLS}} \mu^2 + o(\mu^2)$ uniformly over $(\eta, \Phi) \in D_\mu$.

2. The constrained minimisation problem

We tackle the constrained minimisation problem in two steps. (i) Fix $\eta \neq 0$ and minimise $\mathcal{E}(\eta, \cdot)$ over $T_\mu = \{\Phi \in H_\star^{1/2}(\mathbb{R}) : \mathcal{I}(\eta, \Phi) = 2\mu\}$. This problem (of minimising a quadratic functional over a linear manifold) admits a unique global minimiser Φ_η . (ii) Minimise $\mathcal{J}_\mu(\eta) := \mathcal{E}(\eta, \Phi_\eta)$ over $\eta \in U \setminus \{0\}$.

Because Φ_η minimises $\mathcal{E}(\eta, \cdot)$ over T_μ there exists a Lagrange multiplier ν_η such that $G(\eta)\Phi_\eta = \nu_\eta\eta_x$, and straightforward calculations show that $\Phi_\eta = \nu_\eta G(\eta)^{-1}\eta_x$, $\nu_\eta = \mu/\mathcal{L}(\eta)$ and

$$\mathcal{J}_\mu(\eta) = \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)},$$

where

$$\mathcal{K}(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\eta^2 + \frac{\gamma \eta_{xx}^2}{(1 + \eta_x^2)^{5/2}} \right) dx, \quad \mathcal{L}(\eta) = \frac{1}{2} \int_{-\infty}^{\infty} \eta_x G(\eta)^{-1} \eta_x dx.$$

This computation also shows that the dimensionless speed of the solitary wave corresponding to a constrained minimiser of \mathcal{E} over S_μ is $\mu/\mathcal{L}(\eta)$.

A similar minimisation problem arises in the study of irrotational solitary water waves with weak surface tension (see Groves and Wahlén [4], taking $\omega = 0$ and $\beta < \beta_c$); in that case $\mathcal{K}(\eta)$ is replaced by $\tilde{\mathcal{K}}(\eta) = \int_{-\infty}^{\infty} (\frac{1}{2}\eta^2 + \beta((1 + \eta_x^2)^{1/2} - 1)) dx$. In this note we describe the modifications necessary to apply the theory of Groves and Wahlén to the hydroelastic problem. The presence of the second-order derivative necessitates on the one hand non-trivial modifications because the $L^2(\mathbb{R})$ -gradient $\mathcal{K}'(\eta)$ is not defined on the whole of U , but leads on the other hand to a more satisfactory final result (compare Theorem 1.1 with Theorem 1.5 of Groves & Wahlén).

Lemmata 2.1 and 2.2 state some basic properties of the functionals \mathcal{K} and \mathcal{L} (see Groves and Wahlén [4] for the proof of the latter), while Proposition 2.1 is a useful ‘weak-strong’ argument.

Lemma 2.1

- (i) The functional $\mathcal{K} : H^2(\mathbb{R}) \rightarrow \mathbb{R}$ is analytic at the origin and satisfies $\mathcal{K}(0) = 0$.
- (ii) There exists a constant $D > 0$ such that $\mathcal{K}(\eta) \geq D^{-1}\|\eta\|_2^2$ for all $\eta \in U$.
- (iii) The $L^2(\mathbb{R})$ -gradient $\mathcal{K}'(\eta)$ exists for each $\eta \in U \cap H^4(\mathbb{R})$ and is given by the formula

$$\mathcal{K}'(\eta) = \eta + \gamma \left[\frac{\eta_{xx}}{(1 + \eta_x^2)^{5/2}} \right]_{xx} + \frac{5}{2} \gamma \left[\frac{\eta_x \eta_{xx}^2}{(1 + \eta_x^2)^{7/2}} \right]_x.$$

This formula defines a function $\mathcal{K}' : H^4(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ which is analytic at the origin and satisfies $\mathcal{K}'(0) = 0$.

- (iv) The estimates $|\mathcal{K}_4(\eta)| \lesssim \|\eta\|_2^2 \|\eta\|_{1,\infty}^2$, $|\mathcal{K}_r(\eta)| \lesssim \|\eta\|_2^3 \|\eta\|_{1,\infty}^2$, $|\mathcal{K}_{nl}(\eta)| \lesssim \|\eta\|_{1,\infty}$ hold for all $\eta \in U$, where $\mathcal{K}_n(\eta) = \frac{1}{n!} d^n \mathcal{K}[0](\{\eta\}^n)$, $\mathcal{K}_r(\eta) = \sum_{n=5}^{\infty} \mathcal{K}_n(\eta)$ and $\mathcal{K}_{nl}(\eta) = \mathcal{K}(\eta) - \mathcal{K}_2(\eta)$.
- (v) The estimates

$$\|\mathcal{F}^{-1}[(1 - \chi_S(k))g(k)^{-1/2}\mathcal{F}[\mathcal{K}'_4(\eta)]]\|_0 \lesssim \|\eta\|_2(\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2\eta\|_0)^2,$$

$$\|\mathcal{F}^{-1}[(1 - \chi_S(k))g(k)^{-1/2}\mathcal{F}[\mathcal{K}'_r(\eta)]]\|_0, \quad |\langle \mathcal{K}'_4(\eta), \eta \rangle_0|, \quad |\langle \mathcal{K}'_r(\eta), \eta \rangle_0| \lesssim \|\eta\|_2^2(\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2\eta\|_0)^2$$

hold for all $\eta \in U \cap H^4(\mathbb{R})$, where $S = [-k_0 - \delta_0, -k_0 + \delta_0] \cup [k_0 - \delta_0, k_0 + \delta_0]$ and $\delta_0 \in (0, \frac{1}{3}k_0)$.

Proof. Assertions (i)–(iv) follow by straightforward estimates. Turning to (v), note that

$$\mathcal{K}'_4(\eta) = \frac{5}{2} \gamma ((\eta_x \eta_{xx}^2)_x + (\eta_x^2 \eta_{xx})_{xx}) = \frac{5}{2} \gamma ((\eta_x(\eta_{xx} + k_0^2\eta)^2 - 2k_0^2\eta_x\eta(\eta_{xx} + k_0^2\eta) + k_0^4\eta_x\eta^2)_x + (\eta_x^2\eta_{xx})_{xx})$$

so that

$$\begin{aligned} \|\mathcal{F}^{-1}[(1 - \chi_S(k))g(k)^{-1/2}\mathcal{F}[\mathcal{K}'_4(\eta)]]\|_0 &\lesssim \|\eta_x(\eta_{xx} + k_0^2\eta)^2\|_{-1} + \|\eta_x\eta(\eta_{xx} + k_0^2\eta)\|_0 + \|\eta_x\eta^2\|_0 + \|\eta_x^2\eta_{xx}\|_0 \\ &\lesssim \|\eta_x(\eta_{xx} + k_0^2\eta)\|_0 \|\eta_{xx} + k_0^2\eta\|_0 + \|\eta\|_2 \|\eta\|_{1,\infty}^2 \\ &\lesssim \|\eta\|_2(\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2\eta\|_0)^2, \end{aligned}$$

where we have used the inequalities $(1 - \chi_S(k))g(k)^{-1/2} \gtrsim (1 + |k|^2)^{-1}$ and $\|u_1 u_2\|_{-1} \lesssim \|u_1\|_0 \|u_2\|_0$ (see Hörmander [6, Theorem 8.3.1]); the remaining estimates are obtained in a similar fashion. \square

Lemma 2.2

- (i) The functional $\mathcal{L} : H^{3/2}(\mathbb{R}) \rightarrow \mathbb{R}$ is analytic at the origin and satisfies $\mathcal{L}(0) = 0$.
- (ii) The estimates $\|\eta\|_{1/2}^2 \lesssim \mathcal{L}(\eta)$, $\mathcal{L}_2(\eta) \lesssim \|\eta\|_{1/2}^2$, where $\mathcal{L}_2(\eta) = \frac{1}{2!} d^2 \mathcal{L}[0](\{\eta\}^2)$, hold for all $\eta \in U$.
- (iii) The $L^2(\mathbb{R})$ -gradient $\mathcal{L}'(\eta)$ exists for each $\eta \in U$ and defines a function $\mathcal{L}' : H^2(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ which is analytic at the origin and satisfies $\mathcal{L}'(0) = 0$.
- (iv) Suppose that $\{M_n^{(1)}\}, \{M_n^{(2)}\} \subseteq \mathbb{R}$ and $\{\eta_n^{(1)}\}, \{\eta_n^{(2)}\} \subseteq U$ are sequences with $M_n^{(1)}, M_n^{(2)} \rightarrow \infty$, $M_n^{(1)}/M_n^{(2)} \rightarrow 0$, $\{\eta_n^{(1)} + \eta_n^{(2)}\} \subseteq U$ and $\text{supp } \eta_n^{(1)} \subseteq (-2M_n^{(1)}, 2M_n^{(1)})$, $\text{supp } \eta_n^{(2)} \subseteq \mathbb{R} \setminus (-M_n^{(2)}, M_n^{(2)})$. The functional \mathcal{L} has the ‘pseudolocal’ properties

$$\mathcal{L}(\eta_n^{(1)} + \eta_n^{(2)}) - \mathcal{L}(\eta_n^{(1)}) - \mathcal{L}(\eta_n^{(2)}) \rightarrow 0, \quad \|\mathcal{L}'(\eta_n^{(1)} + \eta_n^{(2)}) - \mathcal{L}'(\eta_n^{(1)}) - \mathcal{L}'(\eta_n^{(2)})\|_0 \rightarrow 0$$

and $\langle \mathcal{L}'(\eta_n^{(2)}), \phi \rangle_0 \rightarrow 0$ for each $\phi \in C_0^\infty(\mathbb{R})$.

- (v) The estimates

$$|\mathcal{L}_3(\eta)| \lesssim \|\eta\|_2^2 (\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2 \eta\|_0), \quad |\mathcal{L}_4(\eta)| \lesssim \|\eta\|_2^2 (\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2 \eta\|_0)^2,$$

$$|\mathcal{L}_r(\eta)| \lesssim \|\eta\|_2^3 (\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2 \eta\|_0)^2, \quad |\mathcal{L}_{nl}(\eta)| \lesssim \|\eta\|_{1,\infty},$$

where $\mathcal{L}_n(\eta) = \frac{1}{n!} d^n \mathcal{L}[0](\{\eta\}^n)$, $\mathcal{L}_r(\eta) = \sum_{n=5}^\infty \mathcal{L}_n(\eta)$ and $\mathcal{L}_{nl}(\eta) = \mathcal{L}(\eta) - \mathcal{L}_2(\eta)$, and

$$\|\mathcal{L}'_3(\eta)\|_0 \lesssim \|\eta\|_2 (\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2 \eta\|_0 + \|K^0 \eta\|_\infty),$$

$$\|\mathcal{L}'_4(\eta)\|_0 \lesssim \|\eta\|_2 (\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2 \eta\|_0 + \|K^0 \eta\|_\infty)^2,$$

$$\|\mathcal{L}'_r(\eta)\|_0 \lesssim \|\eta\|_2^2 (\|\eta\|_{1,\infty} + \|\eta_{xx} + k_0^2 \eta\|_0)^2,$$

where $K_0 \eta := \mathcal{F}^{-1}[f(k)\hat{\eta}]$, hold for all $\eta \in U$.

Proposition 2.1 Suppose that $\{\eta_n\} \subseteq U$ and $\eta \in U$ have the properties that $\eta_n \rightharpoonup \eta$ in $H^2(\mathbb{R})$ and $\eta_n \rightarrow \eta$ in $L^2(\mathbb{R})$ (and hence in $H^s(\mathbb{R})$ for all $s \in [0, 2)$). The inequality $\mathcal{K}(\eta) \leq \lim_{n \rightarrow \infty} \mathcal{K}(\eta_n)$ holds whenever $\{\mathcal{K}(\eta_n)\}$ is convergent, and equality implies that $\eta_n \rightarrow \eta$ in $H^2(\mathbb{R})$.

Proof. Note that $(1 + \eta_{nx}^2)^{-5/4} \eta_{nxx} \rightharpoonup (1 + \eta_x^2)^{-5/4} \eta_{xx}$ in $L^2(\mathbb{R})$, and it follows from the weak lower semicontinuity of $\|\cdot\|_0^2$ (and $\eta_n \rightarrow \eta$ in $L^2(\mathbb{R})$) that $\mathcal{K}(\eta) \leq \lim_{n \rightarrow \infty} \mathcal{K}(\eta_n)$. Moreover, $\mathcal{K}(\eta_n) \rightarrow \mathcal{K}(\eta)$ implies that $\|(1 + \eta_{nx}^2)^{-5/4} \eta_{nxx}\|_0 \rightarrow \|(1 + \eta_x^2)^{-5/4} \eta_{xx}\|_0$, so that $(1 + \eta_{nx}^2)^{-5/4} \eta_{nxx} \rightarrow (1 + \eta_x^2)^{-5/4} \eta_{xx}$ in $L^2(\mathbb{R})$ and hence $\eta_{nxx} \rightarrow \eta_{xx}$ in $L^2(\mathbb{R})$. \square

Finally, we establish some basic properties of \mathcal{J}_μ . The next proposition (cf. Groves and Wahlen [4, Appendix A.2]) shows in particular that $c_\mu := \inf_{\eta \in U \setminus \{0\}} \mathcal{J}_\mu(\eta) < 2\nu_0 \mu$, while Lemma 2.3 shows that its critical points have additional regularity.

Proposition 2.2 The continuous mapping $\alpha \mapsto \nu_0 \mathcal{L}(\eta_\alpha^*)$, where

$$\eta_\alpha^*(x) = \alpha \zeta_{\text{NLS}}(\alpha x) \cos k_0 x - \frac{1}{2} \alpha^2 g(2k_0)^{-1} A_3^1 \zeta_{\text{NLS}}(\alpha x)^2 \cos 2k_0 x - \frac{1}{2} \alpha^2 g(0)^{-1} A_3^2 \zeta_{\text{NLS}}(\alpha x)^2,$$

is invertible, and its (continuous) inverse $\mu \mapsto \alpha(\mu)$ satisfies $\mathcal{J}_\mu(\eta_{\alpha(\mu)}^*) = 2\nu_0 \mu + c_{\text{NLS}} \mu^3 + o(\mu^3)$.

Remark 2 Each $\eta \in U \setminus \{0\}$ satisfies

$$\mathcal{K}_2(\eta) + \frac{\mu^2}{\mathcal{L}_2(\eta)} = \mathcal{K}_2(\eta) - \nu_0^2 \mathcal{L}_2(\eta) + \frac{(\mu - \nu_0 \mathcal{L}_2(\eta))^2}{\mathcal{L}_2(\eta)} + 2\nu_0 \mu \geq \frac{1}{2} \int_{-\infty}^{\infty} g(k) |\hat{\eta}|^2 dk + 2\nu_0 \mu \geq 2\nu_0 \mu.$$

Lemma 2.3 Any critical point $\eta \in U \setminus \{0\}$ of \mathcal{J}_μ belongs to $H^4(\mathbb{R})$.

Proof. Write $u = (1 + \eta_x^2)^{-5/2} \eta_{xx}$, so that $\eta_x (1 + \eta_x^2)^{3/2} u^2 \in L^1(\mathbb{R}) \subseteq H^{-3/4}(\mathbb{R})$, and observe that

$$\gamma u_{xx} = \frac{\mu}{\mathcal{L}(\eta)^2} \mathcal{L}'(\eta) - \eta - \frac{5}{2} \gamma (\eta_x (1 + \eta_x^2)^{3/2} u^2)_x \quad (7)$$

in the sense of distributions since η is a critical point of \mathcal{J}_μ . It follows from (7) and the fact that $\mathcal{L}'(\eta) \in L^2(\mathbb{R})$ that $\gamma u_{xx} \in H^{-7/4}(\mathbb{R})$, that is $u \in H^{1/4}(\mathbb{R})$. We conclude that $u^2 \in L^2(\mathbb{R})$ (see Hörmander [6, Theorem 8.3.1]), so that $\eta_x (1 + \eta_x^2)^{3/2} u^2 \in L^2(\mathbb{R})$ and hence $\gamma u_{xx} \in H^{-1}(\mathbb{R})$, that is $u \in H^1(\mathbb{R})$.

Observing that $\eta_x (1 + \eta_x^2)^{3/2} u^2 \in H^1(\mathbb{R})$, one finds from (7) that $\gamma u_{xx} \in L^2(\mathbb{R})$, $u \in H^2(\mathbb{R})$ and finally $\eta \in H^4(\mathbb{R})$ (because $\eta_{xx} = (1 + \eta_x^2)^{5/2} u$). \square

3. A special minimising sequence

Any function $\eta \in U$ with $\mathcal{J}_\mu(\eta) < 2\nu_0\mu$ satisfies $\|\eta\|_2^2 < 2D\nu_0\mu$, $\mathcal{L}(\eta) > \mu/(2\nu_0)$ and $\mathcal{L}_2(\eta) \gtrsim \mu$ (see Lemmata 2.1(ii) and 2.2(ii)). These properties are enjoyed in particular by a minimising sequence $\{\eta_n\}$ for \mathcal{J}_μ over $U \setminus \{0\}$, which also satisfies $\mathcal{M}_\mu(\eta_n) \lesssim -\mu^3$, where $\mathcal{M}_\mu(\eta) = \mathcal{J}_\mu(\eta) - \mathcal{K}_2(\eta) - \mu^2/\mathcal{L}_2(\eta)$ (Proposition 2.2), and hence $\|\eta_n\|_{1,\infty} \gtrsim \mu^3$ (because $|\mathcal{K}_{nl}(\eta_n)|, |\mathcal{L}_{nl}(\eta_n)| \lesssim \|\eta_n\|_{1,\infty}$). In this section we construct a special minimising sequence $\{\tilde{\eta}_n\}$ for \mathcal{J}_μ over $U \setminus \{0\}$ which has additional features.

Theorem 3.1 *There exists a minimising sequence $\{\tilde{\eta}_n\}$ for \mathcal{J}_μ over $U \setminus \{0\}$ with the properties that $\{\tilde{\eta}_n\} \subseteq H^4(\mathbb{R})$ and $\|\mathcal{J}'_\mu(\tilde{\eta}_n)\|_0 \rightarrow 0$.*

The first step in the proof of this theorem is a recursive application of the concentration-compactness principle to the sequence $\{\eta_{nx}^2 + \eta_n^2\} \subseteq L^1(\mathbb{R})$, where $\{\eta_n\}$ is an arbitrary minimising sequence for \mathcal{J}_μ over $U \setminus \{0\}$; the procedure is a straightforward modification of that used by Buffoni *et al.* [2] for a penalised minimisation problem.

‘*Vanishing*’ is excluded since it implies that $\|\eta_n\|_{1,\infty} \rightarrow 0$.

‘*Concentration*’ implies the existence of $\eta \in U$ with $\eta_n \rightharpoonup \eta$ in $H^2(\mathbb{R})$ and $\eta_n \rightarrow \eta$ in $L^2(\mathbb{R})$ (up to subsequences and translations). Since $\mathcal{K}(\eta_n) \leq \mathcal{J}_\mu(\eta_n) < 2\nu_0\mu$ the sequence $\{\mathcal{K}(\eta_n)\}$ is bounded and hence admits a convergent subsequence (still denoted by $\{\mathcal{K}(\eta_n)\}$) which satisfies $\mathcal{K}(\eta) \leq \lim_{n \rightarrow \infty} \mathcal{K}(\eta_n)$ (Proposition 2.1). Lemma 2.2(i) asserts that $\mathcal{L}(\eta_n) \rightarrow \mathcal{L}(\eta)$, so that $\mathcal{J}_\mu(\eta) \leq \lim_{n \rightarrow \infty} \mathcal{J}(\eta_n) = c_\mu$, which therefore holds with equality; it follows that $\mathcal{K}(\eta_n) \rightarrow \mathcal{K}(\eta)$ and hence $\eta_n \rightarrow \eta$ in $H^2(\mathbb{R})$ (Proposition 2.1), so that η minimises \mathcal{J}_μ over $U \setminus \{0\}$.

‘*Dichotomy*’ leads to the existence of sequences $\{\eta_n^{(1)}\}, \{\eta_n^{(2)}\}$ of the kind described in Lemma 2.2(iv) with $\lim_{n \rightarrow \infty} \|\eta_n - \eta_n^{(1)} - \eta_n^{(2)}\|_2 = 0$ (up to subsequences and translations), so that in particular

$$\lim_{n \rightarrow \infty} \mathcal{J}_\mu(\eta_n) = \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(1)}}(\eta_n^{(1)}) + \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(2)}}(\eta_n^{(2)}),$$

where $\mu^{(j)} = \mu \lim_{n \rightarrow \infty} \mathcal{L}(\eta_n^{(j)}) / \lim_{n \rightarrow \infty} \mathcal{L}(\eta_n)$ (so that $\mu^{(1)} + \mu^{(2)} = \mu$). Furthermore the sequence $\{\eta_n^{(1)}\}$ exhibits ‘concentration’, and for every $\{v_n\} \subseteq U$ with $\{\eta^{(1)} + v_n\} \subseteq U$ there exists an increasing, unbounded sequence $\{S_n\}$ of positive real numbers such that

$$\lim_{n \rightarrow \infty} \mathcal{J}_\mu(\eta^{(1)} + \tau_{S_n} v_n) \leq \mathcal{J}_{\mu^{(1)}}(\eta^{(1)}) + \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(2)}}(v_n),$$

where $(\tau_X v_n)(x) := v_n(x + X)$ (see Buffoni *et al.* [2, Proposition 3.14]). Arguing as above, we find that $\mathcal{K}(\eta^{(1)}) \leq \lim_{n \rightarrow \infty} \mathcal{K}(\eta_n^{(1)})$ and $\mathcal{L}(\eta^{(1)}) = \lim_{n \rightarrow \infty} \mathcal{L}(\eta_n^{(1)})$, so that $\mathcal{J}_{\mu^{(1)}}(\eta^{(1)}) \leq \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(1)}}(\eta_n^{(1)})$. This estimate is a contradiction unless equality holds because $\lim_{n \rightarrow \infty} \mathcal{J}_\mu(\eta^{(1)} + \tau_{S_n} \eta_n^{(2)}) \leq \mathcal{J}_{\mu^{(1)}}(\eta^{(1)}) + \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(2)}}(\eta_n^{(2)})$. One concludes that $\mathcal{K}(\eta_n^{(1)}) \rightarrow \mathcal{K}(\eta^{(1)})$ and hence $\eta_n^{(1)} \rightarrow \eta^{(1)}$ in $H^2(\mathbb{R})$.

Without loss of generality we may assume that $\{\eta_n\}$ is a Palais-Smale sequence for \mathcal{J}_μ , so that $d\mathcal{J}_\mu[\eta_n] = d\mathcal{K}[\eta_n] - (\mu/\mathcal{L}(\eta_n))^2 d\mathcal{L}[\eta_n] \rightarrow 0$ in $(H^2(\mathbb{R}))^*$ and hence

$$d\mathcal{K}[\eta_n^{(1)}](\phi) + d\mathcal{K}[\eta_n^{(2)}](\phi) - \frac{\mu^2}{\mathcal{L}(\eta_n)^2} \left(\langle \mathcal{L}'(\eta_n^{(1)}), \phi \rangle_0 + \langle \mathcal{L}'(\eta_n^{(2)}), \phi \rangle_0 \right) \rightarrow 0$$

for each $\phi \in C_0^\infty(\mathbb{R})$, where we have used the explicit formula

$$d\mathcal{K}[\eta](\phi) = \int_{-\infty}^{\infty} \left(\eta\phi + \gamma \frac{\eta_{xx}\phi_{xx}}{(1+\eta_x^2)^{5/2}} - \frac{5}{2} \gamma \frac{\eta_x \eta_{xx}^2 \phi_x}{(1+\eta_x^2)^{7/2}} \right) dx$$

to deduce that $d\mathcal{K}[\eta_n^{(1)} + \eta_n^{(2)}](\phi) = d\mathcal{K}[\eta_n^{(1)}](\phi) + d\mathcal{K}[\eta_n^{(2)}](\phi)$. Taking the limit $n \rightarrow \infty$ (and noting that $d\mathcal{K}[\eta_n^{(2)}](\phi) = 0$ for sufficiently large values of n), we find that $d\mathcal{J}_{\mu^{(1)}}[\eta^{(1)}](\phi) = d\mathcal{K}[\eta^{(1)}](\phi) - (\mu^{(1)}/\mathcal{L}(\eta)) d\mathcal{L}[\eta^{(1)}](\phi) = d\mathcal{K}[\eta^{(1)}](\phi) - (\mu^{(1)}/\mathcal{L}(\eta))^2 \langle \mathcal{L}'(\eta^{(1)}), \phi \rangle_0 = 0$ for each $\phi \in C_0^\infty(\mathbb{R})$.

This argument shows that $\eta^{(1)}$ is a critical point of $\mathcal{J}_{\mu^{(1)}}$, while $\{\eta_n^{(2)}\}$ is a minimising sequence for $\mathcal{J}_{\mu_2} : U_2 \setminus \{0\} \rightarrow \mathbb{R}$ given by $\mathcal{J}_{\mu_2}(\eta) = \mathcal{K}(\eta) + \mu_2^2/\mathcal{L}(\eta)$, where $U_2 = \{\eta \in H^2(\mathbb{R}) : \|\eta\|_2^2 < M^2 - \|\eta^{(1)}\|_2^2\}$

and $\mu_2 = \mu^{(2)}$ (the existence of a minimising sequence $\{v_n\}$ for \mathcal{J}_{μ_2} over $U_2 \setminus \{0\}$ with $\lim_{n \rightarrow \infty} \mathcal{J}_{\mu_2}(v_n) < \lim_{n \rightarrow \infty} \mathcal{J}_{\mu_2}(\eta_n^{(2)})$) leads to the contradiction

$$\lim_{n \rightarrow \infty} \mathcal{J}_{\mu}(\eta^{(1)} + \tau_{S_n} v_n^{(2)}) \leq \mathcal{J}_{\mu^{(1)}}(\eta^{(1)}) + \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(2)}}(v_n^{(2)}) < \mathcal{J}_{\mu^{(1)}}(\eta^{(1)}) + \lim_{n \rightarrow \infty} \mathcal{J}_{\mu^{(2)}}(\eta_n^{(2)}) = c_{\mu}.$$

Lemma 3.2 *There exist $m \in \mathbb{N} \cup \{\infty\}$, a sequence $\{\eta^{(k)}\}_{k=1}^m \subseteq H^4(\mathbb{R})$ and monotone decreasing sequences $\{\mu_k\}_{k=1}^m$, $\{\mu_k^{(1)}\}_{k=1}^m$ of positive real numbers with the properties that*

- (i) $\eta^{(k)}$ is a critical point of the functional $\mathcal{J}_{\mu_k^{(1)}} : U_k \setminus \{0\} \rightarrow \mathbb{R}$ defined by $\mathcal{J}_{\mu_k^{(1)}}(\eta) = \mathcal{K}(\eta) + (\mu_k^{(1)})^2 / \mathcal{L}(\eta)$, where $U_k = \left\{ \eta \in H^2(\mathbb{R}) : \|\eta\|_2^2 < M^2 - \sum_{j=1}^{k-1} \|\eta^{(j)}\|_2^2 \right\}$ and $\mu_1 = \mu$, $U_1 = U$ (so that $\eta^{(k)} \in H^4(\mathbb{R})$ by Lemma 2.3),
- (ii) $\mu_k^{(1)} \in (0, \mu_k)$ for $k = 1, \dots, m-1$ and $\mu_m^{(1)} = \mu_m$ if $m < \infty$,
- (iii) $c_{\mu} = \sum_{k=1}^m \mathcal{J}_{\mu_k^{(1)}}(\eta^{(k)})$,
- (iv) $0 < \|\eta^{(k)}\|_2^2 \leq D\mathcal{K}(\eta^{(k)})$ for $k = 1, \dots, m$ (so that in particular

$$\sum_{k=1}^m \|\eta^{(j)}\|_2^2 \leq D \sum_{k=1}^m \mathcal{K}(\eta^{(k)}) \leq D \sum_{k=1}^m \mathcal{J}_{\mu_k^{(1)}}(\eta^{(k)}) \leq Dc_{\mu} < 2D\nu_0\mu \Big).$$

The special minimising sequence advertised in Theorem 3.1 is constructed from $\{\eta^{(k)}\}_{k=1}^m$ using the following algorithm (see Buffoni *et al.* [2, §3.4]): (i) choose $R_j > 1$ large enough so that $\|\eta^{(j)}\|_{H^2(|x| > R_j)} < 2^{-j}\mu$; (ii) write $S_1 = 0$ and choose $S_j > S_{j-1} + 2R_j + 2R_{j-1}$ for $j = 2, \dots, k$; (iii) define $\tilde{\eta}_n = \sum_{j=1}^m \tau_{S_j + (j-1)n} \eta^{(j)}$.

4. Existence theory

Theorem 1.1 is a consequence of the following result (cf. Groves & Wahlén [4, Theorem 5.2]), which is proved by applying the concentration-compactness principle to the sequence $\{\eta_{n,x}^2 + \eta_n^2\} \subseteq L^1(\mathbb{R})$; ‘vanishing’ and ‘dichotomy’ are readily excluded using the estimate $\|\eta_n\|_{1,\infty} \gtrsim \mu^3$ (see above) and the fact that c_{μ} is a strictly sub-additive function of μ (which is established below).

Theorem 4.1 *The following statements hold for all sufficiently small values of k_0 .*

- (i) *The set B_{μ} of minimisers of \mathcal{J}_{μ} over $U \setminus \{0\}$ is nonempty and lies in $H^4(\mathbb{R})$. Moreover, each $\eta \in B_{\mu}$ satisfies $\|\eta\|_2^2 \leq 2D\nu_0\mu$.*
- (ii) *Suppose that $\{\eta_n\}$ is a minimising sequence for \mathcal{J}_{μ} over $U \setminus \{0\}$. There exists a sequence $\{x_n\} \subseteq \mathbb{R}$ with the property that there exists a subsequence of $\{\eta_n(x_n + \cdot)\}$ which converges in $H^2(\mathbb{R})$ to a function $\eta \in B_{\mu}$.*

We use the special minimising sequence $\{\tilde{\eta}_n\}$ to show that c_{μ} is a strictly sub-additive function of μ . We begin by deriving sharper estimates for a ‘near minimiser’ of \mathcal{J}_{μ} over $U \setminus \{0\}$, that is a function $\tilde{\eta} \in U \setminus \{0\}$ with $\tilde{\eta} \in H^4(\mathbb{R})$, $\|\mathcal{J}'_{\mu}(\tilde{\eta})\|_0 \leq \mu^N$ for some $N \in \mathbb{N}$ and $\mathcal{J}_{\mu}(\tilde{\eta}) < 2\nu_0\mu$ (and hence $\|\tilde{\eta}\|_2 \lesssim \mu^{1/2}$, $\mathcal{L}(\tilde{\eta}), \mathcal{L}_2(\tilde{\eta}) \geq \mu$); these estimates apply in particular to the special minimising sequence $\{\tilde{\eta}_n\}$.

We write the equation $\mathcal{J}'_{\mu}(\eta) = \mathcal{K}'(\eta) - (\mu/\mathcal{L}(\eta))^2 \mathcal{L}'(\eta)$ (for $\eta \in U \cap H^4(\mathbb{R})$) in the form

$$g(k)\tilde{\eta} = \mathcal{F} \left[\mathcal{J}'_{\mu}(\eta) - \mathcal{K}'_{\text{nl}}(\eta) + \left(\frac{\mu}{\mathcal{L}(\eta)} + \nu_0 \right) \left(\frac{\mu}{\mathcal{L}(\eta)} - \nu_0 \right) \mathcal{L}'_2(\eta) + \left(\frac{\mu}{\mathcal{L}(\eta)} \right)^2 \mathcal{L}'_{\text{nl}}(\eta) \right]$$

and decompose it into two coupled equations by defining $\eta_2 \in H^4(\mathbb{R})$ by the formula

$$\eta_2 = \mathcal{F}^{-1} \left[\frac{1 - \chi_S(k)}{g(k)} \mathcal{F} \left[\mathcal{J}'_{\mu}(\eta) - \mathcal{K}'_{\text{nl}}(\eta) + \left(\frac{\mu}{\mathcal{L}(\eta)} + \nu_0 \right) \left(\frac{\mu}{\mathcal{L}(\eta)} - \nu_0 \right) \mathcal{L}'_2(\eta) + \left(\frac{\mu}{\mathcal{L}(\eta)} \right)^2 \mathcal{L}'_{\text{nl}}(\eta) \right] \right]$$

(recall that $(1 - \chi_S(k))g(k)^{-1/2} \gtrsim (1 + |k|^2)^{-1}$) and $\eta_1 \in H^2(\mathbb{R})$ by $\eta_1 = \eta - \eta_2$, so that $\text{supp } \hat{\eta}_1 \in S$ and $\chi_S \mathcal{L}'_3(\eta_1) = 0$ (see Groves and Wahlén [4, Proposition 4.15]). We accordingly write these equations as

$$g(k)\hat{\eta}_1 = \chi_S(k)\mathcal{F}[\mathcal{R}(\eta) - \mathcal{K}'_{\text{nl}}(\eta)], \quad \eta_3 := \eta_2 + H(\eta) = \mathcal{F}^{-1} \left[\frac{1 - \chi_S(k)}{g(k)} \mathcal{F}[\mathcal{R}(\eta) - \mathcal{K}'_{\text{nl}}(\eta)] \right],$$

where

$$H(\eta) := \mathcal{F}^{-1} \left[\frac{1}{g(k)} \mathcal{F} \left[- \left(\frac{\mu}{\mathcal{L}(\eta)} \right)^2 \mathcal{L}'_3(\eta_1) \right] \right],$$

$$\mathcal{R}(\eta) := \mathcal{J}'_\mu(\eta) + \left(\frac{\mu}{\mathcal{L}(\eta)} + \nu_0 \right) \left(\frac{\mu}{\mathcal{L}(\eta)} - \nu_0 \right) \mathcal{L}'_2(\eta) + \left(\frac{\mu}{\mathcal{L}(\eta)} \right)^2 (\mathcal{L}'_{\text{nl}}(\eta) - \mathcal{L}'_3(\eta_1)).$$

The next step is to study η_1 using the scaled norm

$$\|\eta_1\|_\alpha := \left(\int_{-\infty}^{\infty} (1 + \mu^{-4\alpha}(|k| - k_0)^4) |\hat{\eta}_1(k)|^2 dk \right)^{1/2}$$

for $H^2(\mathbb{R})$; we choose $\alpha > 0$ as large as possible so that $\|\tilde{\eta}_1\|_\alpha \lesssim \mu^{1/2}$.

Lemma 4.2 *Each near minimiser $\tilde{\eta}$ of \mathcal{J}_μ over $U \setminus \{0\}$ satisfies $\|H(\tilde{\eta})\|_2 \lesssim \mu^{1+\alpha/2} \|\tilde{\eta}_1\|_\alpha$, $\|\mathcal{R}(\tilde{\eta})\|_0 \lesssim \mu^{1/2+2\alpha} \|\tilde{\eta}_1\|_\alpha^2 + \mu^{2N}$ and $\|\mathcal{F}^{-1}[(1 - \chi_S(k))g(k)^{-1/2} \mathcal{F}[\mathcal{K}'_{\text{nl}}(\tilde{\eta})]]\|_0 \lesssim \mu^{1/2+\alpha} \|\tilde{\eta}_1\|_\alpha^2 + \mu \|\tilde{\eta}_3\|_2$.*

Proof. The results for $H(\tilde{\eta})$ and $\mathcal{R}(\tilde{\eta})$ were derived by Groves & Wahlén [4, §4.3.1], while that for $\mathcal{K}'_{\text{nl}}(\tilde{\eta})$ follows from Lemma 2.1(v) and the estimates $\|\eta_1\|_{1,\infty} \lesssim \mu^{\frac{3}{2}} \|\eta_1\|_\alpha$ and $\|\eta_{1xx} + k_0^2 \eta_1\|_0 \leq c\mu^\alpha \|\eta_1\|_\alpha$ (Groves and Wahlén [4, Proposition 4.1]). \square

Square integrating the equation $g(k)\hat{\eta}_1 = \chi(k)\mathcal{F}[\mathcal{R}(\eta) - \mathcal{K}'_{\text{nl}}(\eta)]$, multiplying by $\mu^{-4\alpha}$ and adding $\|\tilde{\eta}_1\|_0^2 \lesssim \mu$ yields $\|\tilde{\eta}_1\|_\alpha^2 \lesssim \mu^{1-2\alpha} \|\tilde{\eta}_1\|_\alpha^4 + \mu$, which implies that $\|\tilde{\eta}_1\|_\alpha^2 \lesssim \mu$ for each $\alpha < 1$; it follows that $\|\tilde{\eta}_3\|_2^2 \lesssim \mu^{3+2\alpha}$ and $\|H(\tilde{\eta})\|_2^2 \lesssim \mu^{2+\alpha}$ for each $\alpha < 1$. These estimates are used to establish the following proposition (see Groves & Wahlén [4, §4.3.2]).

Proposition 4.1 *Suppose that $\tilde{\eta}$ is a near minimiser of \mathcal{J}_μ over $U \setminus \{0\}$. The estimates*

$$\mathcal{M}_{a^2\mu}(a\tilde{\eta}) = -a^3\nu_0^2\mathcal{L}_3(\tilde{\eta}) - a^4\nu_0^2\mathcal{L}_3(\tilde{\eta}) + a^3o(\mu^3),$$

$$\langle \mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta} \rangle_0 + 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) = -3a^3\nu_0^2\mathcal{L}_3(\tilde{\eta}) - 4a^4\nu_0^2\mathcal{L}_3(\tilde{\eta}) + a^3o(\mu^3),$$

where $\tilde{\mathcal{M}}_\mu(\eta) = \mu/\mathcal{L}(\eta) - \mu/\mathcal{L}_2(\eta)$, hold uniformly over $a \in [1, 2]$.

Lemma 4.3 *Each near minimiser $\tilde{\eta}$ of \mathcal{J}_μ over $U \setminus \{0\}$ satisfies the estimate*

$$\mathcal{K}_4(\tilde{\eta}) = A_4^1 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + o(\mu^3).$$

Proof. We expand the right-hand side of the formula

$$\mathcal{K}_4(\tilde{\eta}) = -\frac{5}{4}\gamma \int_{-\infty}^{\infty} (\partial_x(\tilde{\eta}_1 + H(\tilde{\eta}) + \tilde{\eta}_3))^2 \partial_x^2((\tilde{\eta}_1 + H(\tilde{\eta}) + \tilde{\eta}_3))^2 dx;$$

terms with zero, one or two occurrences of $\tilde{\eta}_1$ are $O((\|\tilde{\eta}_1\|_2 + \|H(\tilde{\eta})\|_2 + \|\tilde{\eta}_3\|)^2(\|H(\tilde{\eta})\|_2 + \|\tilde{\eta}_3\|)^2)$ and hence $O(\mu\mu^{2+\alpha}) = o(\mu^3)$, while terms with three occurrences of $\tilde{\eta}_1$ are estimated by $O((\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_{1xx} + k_0^2\tilde{\eta}_1\|_0)\|\tilde{\eta}_1\|_2^2(\|H(\tilde{\eta})\|_2 + \|\tilde{\eta}_3\|)^2) = O(\mu^{2+\alpha}\|\tilde{\eta}_1\|) = O(\mu^{5/2+\alpha}) = o(\mu^3)$, so that $\mathcal{K}_4(\tilde{\eta}) = -\frac{5}{4}\gamma \int_{-\infty}^{\infty} \tilde{\eta}_{1x}^2 \tilde{\eta}_{1xx}^2 dx + o(\mu^3)$.

Writing $\tilde{\eta}_1 = \tilde{\eta}_1^+ + \tilde{\eta}_1^-$, where $\tilde{\eta}_1^+ = \mathcal{F}^{-1}[\chi_{[0,\infty)}\mathcal{F}[\tilde{\eta}_1]]$, $\tilde{\eta}_1^- = \mathcal{F}^{-1}[\chi_{(-\infty,0]}\mathcal{F}[\tilde{\eta}_1]]$, we find that

$$\|(ik \mp ik_0)\tilde{\eta}_1^\pm\|_s^2 = \|(|k| - k_0)\mathcal{F}[\tilde{\eta}_1]\|_0^2 \leq \frac{1}{2} \int_{-\infty}^{\infty} (\mu^{2\alpha} + \mu^{-2\alpha}(|k| - k_0)^4) |\mathcal{F}[\tilde{\eta}_1]|^2 dk \lesssim \mu^{2\alpha} \|\tilde{\eta}_1\|^2 \lesssim \mu^{1+2\alpha}$$

so that $(\tilde{\eta}_1^\pm)_x = \pm ik_0 + O(\mu^{1+2\alpha})$ in $H^s(\mathbb{R})$ for each $s \geq 0$. Using this estimate, one concludes that

$$\int_{-\infty}^{\infty} \tilde{\eta}_{1x}^2 \tilde{\eta}_{1xx}^2 dx = \int_{-\infty}^{\infty} ((\tilde{\eta}_{1x}^+)^2 (\tilde{\eta}_{1xx}^-)^2 + (\tilde{\eta}_{1x}^-)^2 (\tilde{\eta}_{1xx}^+)^2 + 4\tilde{\eta}_{1x} \tilde{\eta}_{1xx} \tilde{\eta}_{1xx}^+ \tilde{\eta}_{1xx}^-) dx$$

$$\begin{aligned}
&= 2k_0^6 \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2 (\tilde{\eta}_1^-)^2 dx + o(\mu) \\
&= \frac{1}{3} k_0^2 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + o(\mu). \quad \square
\end{aligned}$$

The corresponding estimates for $\mathcal{L}_3(\tilde{\eta})$ and $\mathcal{L}_4(\tilde{\eta})$ are derived similarly by Groves and Wahlén [4, §4.3.2].

Lemma 4.4 *Each near minimiser $\tilde{\eta}$ of \mathcal{J}_μ over $U \setminus \{0\}$ satisfies the estimates*

$$-\nu_0^2 \mathcal{L}_3(\tilde{\eta}) = A_3 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + o(\mu^3), \quad \mathcal{L}_4(\tilde{\eta}_1) = A_4^2 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + o(\mu^3).$$

Corollary 4.5 *Suppose that $\tilde{\eta}$ is a near minimiser of \mathcal{J}_μ over $U \setminus \{0\}$. The estimates*

$$\begin{aligned}
\mathcal{M}_{a^2\mu}(a\tilde{\eta}) &= (a^3 A_3 + a^4 A_4) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + a^3 o(\mu^3), \\
\langle \mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta} \rangle_0 + 4a^2 \mu \tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) &= (3a^3 A_3 + 4a^4 A_4) \int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx + a^3 o(\mu^3)
\end{aligned}$$

hold uniformly over $a \in [1, 2]$, and $\int_{-\infty}^{\infty} \tilde{\eta}_1^4 dx \gtrsim \mu^3$.

Lemma 4.6 *Suppose that $\tilde{\eta}$ is a near minimiser of \mathcal{J}_μ over $U \setminus \{0\}$. For each sufficiently small value of k_0 the function $a \mapsto a^{-5/2} \mathcal{M}_{a^2\mu}(a\tilde{\eta})$, $a \in [1, 2]$ is decreasing and strictly negative.*

Proof. Since $A_3 \leq 0$ and $\lim_{k_0 \rightarrow 0} A_4 = -\frac{1}{2}$, so that $A_4 \leq 0$ for sufficiently small values of k_0 , we have

$$\begin{aligned}
\frac{d}{da} \left(a^{-5/2} \mathcal{M}_{a^2\mu}(a\tilde{\eta}) \right) &= a^{-7/2} \left(-\frac{5}{2} \mathcal{M}_{a^2\mu}(a\tilde{\eta}) + \langle \mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta} \rangle_0 + 4a^2 \mu \tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) \right) \\
&\leq \frac{1}{2} a^{-1/2} \left((A_3 + 6A_4) \int_{\mathbb{R}} \tilde{\eta}_1^4 dx + o(\mu^3) \right) \\
&\lesssim -\mu^3 + o(\mu^3) \\
&< 0. \quad \square
\end{aligned}$$

Corollary 4.7 *For each sufficiently small value of k_0 the strict sub-homogeneity criterion $c_{a\mu} < ac_\mu$ holds for each $a > 1$ (so that in particular c_μ is a strictly sub-additive function of μ).*

Proof. It suffices to prove this inequality for $a \in (1, 4]$. Replacing a by $a^{1/2}$, we find from Lemma 4.6 that $\mathcal{M}_{a\mu}(a^{1/2}\tilde{\eta}_n) \leq a^{5/4} \mathcal{M}_\mu(\tilde{\eta}_n)$ and therefore that

$$c_{a\mu} \leq \mathcal{J}_{a\mu}(\tilde{\eta}_n) \leq a \left(\mathcal{K}_2(\tilde{\eta}_n) + \frac{\mu^2}{\mathcal{L}_2(\tilde{\eta}_n)} \right) + a^{5/4} \mathcal{M}_\mu(\tilde{\eta}_n) = a \mathcal{J}_\mu(\tilde{\eta}_n) + (a^{5/4} - a) \mathcal{M}_\mu(\tilde{\eta}_n)$$

for $a \in (1, 4]$. In the limit $n \rightarrow \infty$ the above inequality yields $c_{a\mu} < ac_\mu$ since $\limsup_{n \rightarrow \infty} \mathcal{M}_\mu(\tilde{\eta}_n) < 0$. \square

Remark 3 *Theorem 1.2 is proved by Groves & Wahlén [4, §5.2.2]; the proof additionally confirms a posteriori that the estimates $\|\tilde{\eta}_1\|_\alpha^2 \lesssim \mu$, $\|\tilde{\eta}_3\|_2^2 \lesssim \mu^{3+2\alpha}$ and $\|H(\tilde{\eta})\|_2^2 \lesssim \mu^{2+\alpha}$ also hold for $\alpha = 1$.*

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References

- [1] D. M. Ambrose, M. Siegel, Well-posedness of two-dimensional hydroelastic waves, preprint (2014).
- [2] B. Buffoni, M. D. Groves, S. M. Sun, E. Wahlén, Existence and conditional energetic stability of three-dimensional fully localised solitary gravity-capillary water waves, J. Diff. Eqns. 254 (2013) 1006–1096.
- [3] F. Dias, C. Kharif, Nonlinear gravity and capillary-gravity waves, Ann. Rev. Fluid Mech. 31 (1999) 301–346.
- [4] M. D. Groves, E. Wahlén, Existence and conditional energetic stability of solitary gravity-capillary water waves with constant vorticity, Proc. Roy. Soc. Edin. A 145 (2015) 791–883.
- [5] P. Guyenne, E. Parau, Computations of fully nonlinear hydroelastic solitary waves on deep water, J. Fluid Mech. 713 (2012) 307–329.
- [6] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Heidelberg: Springer-Verlag (1997).
- [7] P. A. Milewski, Z. Wang, Three dimensional flexural-gravity waves, Stud. Appl. Math. 131 (2013) 135–148.
- [8] P. Plotnikov, J. F. Toland, Modelling nonlinear hydroelastic waves, Phil. Trans. R. Soc. Lond. A 369 (2011) 2942–2956.